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Internal Memorandum

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PART II

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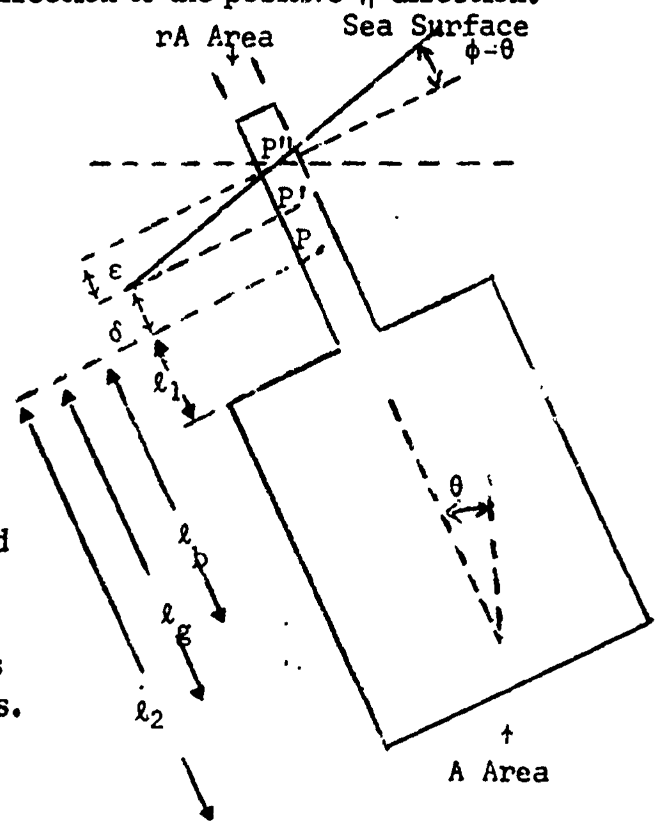
Equations of motion are given below which take account of all the pressure forces with high accuracy, but completely neglect drag forces. The results of the previous simplified study of the vertical motion are confirmed; the principal new result is the amount of the horizontal force and its torque. Nothing large or important is seen in the interaction terms.

The primary dependent variables are ξ, η, and θ. The first two are displacements of the center of mass from its equilibrium position in still water, respectively on a horizontal line along the direction of wave propagation, and vertically. Positive directions are respectively with the wave travel, and upward. θ is an angle of rotation about a horizontal axis parallel to the wave fronts. Positive sense for θ is from the positive ξ direction to the positive η direction.

<u>Notation:</u> Upper and lower cross-sections are rA and A. Axial positions are measured from point P which is at the waterline when the buoy is at rest in still water. Meanings of ℓ_1 and ℓ_2 are indicated in the diagram; ℓ_b and ℓ_g give respective positions of center of buoyancy (for immersion in still water to point P) and center of mass.

Mass of the buoy is expressed as ρ AL, where ρ is density of the displaced sea water. We have $L = \ell_2 - (1 - r)\ell_1$. R is the radius of gyration about the center of mass.



P' is the instantaneous intersection of the buoy axis with the undisturbed sea surface and P" its intersection with the actual sea surface. P' and P" move with respect to the buoy, while P is fixed. δ and ε measure the displacements, positive when upward, PP' and P'P" respectively. ϕ is the inclination of the sea surface at P" from the horizontal, with the same positive sense as θ.

Description of wave motion: Eulerian positions x and y will be referred to the same coordinate directions as ξ and η, respectively. The undisturbed sea surface is $y = 0$. The waves will be assumed to be unidirectional, in the positive direction, with surface amplitude a and single radian frequency ω. The corresponding radian wave number k is given by $\omega^2 = gk$ with g the acceleration of gravity. Particle displacements x' and y' are then given as functions of position (x, y) and time t by

$$x' = ae^{ky} \cos (kx - \omega t)$$

$$y' = ae^{ky} \sin (kx - \omega t) \qquad (1)$$

The usual expression taken for the pressure p is

$$p = \rho g \left[-y + ae^{ky} \sin (kx - \omega t) \right] \qquad (2)$$

(1) and (2) are linear approximations; the limitation in accuracy can be seen by noting that from (1) the height of the sea surface is

$$y_o' = a \sin (kx - \omega t)$$

whereas, from (2) the locus of vanishing pressure is the solution of

$$y = ae^{ky} \sin (kx - \omega t)$$

These agree only if a factor as large as e^{ka} is ignored in the exponential. Hence (2) may be and was replaced by

$$p = \rho g \left[-y + ae^{k(y - y_o')} \sin (kx - \omega t) \right] \qquad (3)$$

which vanishes accurately at the sea surface. Components of the pressure gradient were taken as

$$\frac{\partial p}{\partial x} = \rho\, g a k e^{k(y - y_o')} \cos (kx - \omega t)$$

$$\frac{\partial p}{\partial y} = \rho\, g[-1 + a k e^{k(y - y_o')} \sin (kx - \omega t)] \qquad (4)$$

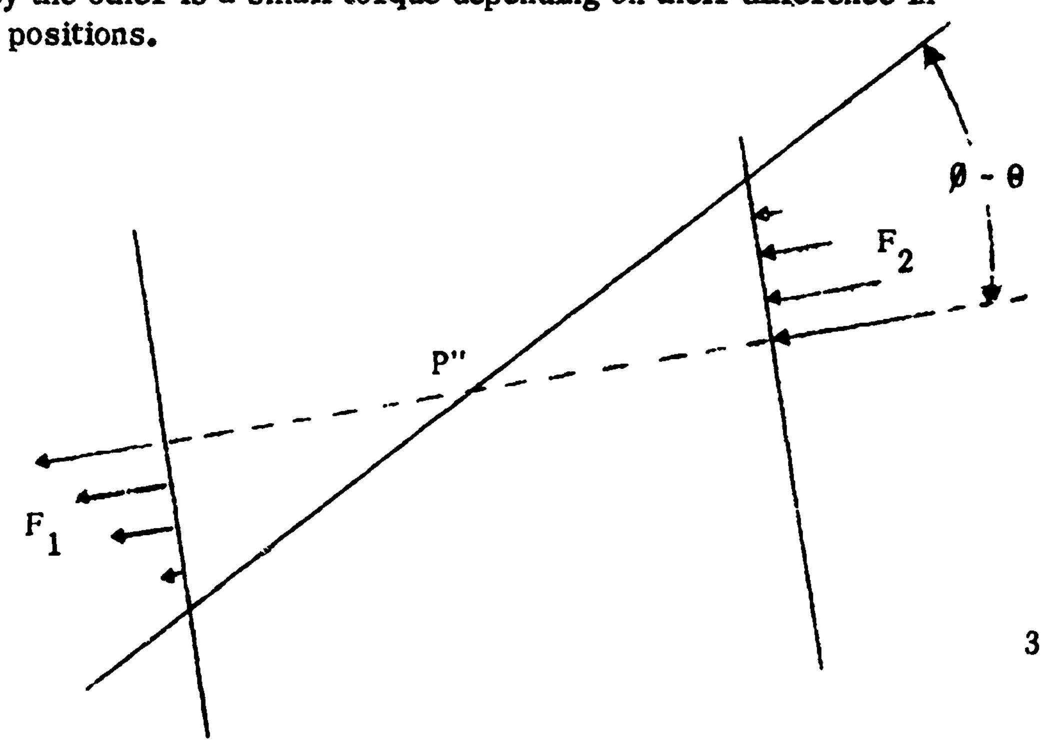
The value of the phase $kx - \omega t$ at point P'' is abbreviated to α. We also have

$$\delta = \ell g (\sec \theta - 1) - \eta \sec \theta$$

$$\epsilon = a \sec \theta \cos \alpha$$

$$\phi = ka \cos \alpha$$

General description of calculation: The lateral pressure force on a volume element of the buoy whose base area is A and whose axial extent is $d\ell$ was taken to be the product of the volume $Ad\ell$ and the component of the pressure gradient (4) on the buoy axis and normal to it. Total lateral force or its moment was calculated by integration of such elements between P'' and the bottom of the buoy. The fact that the sea surface was not necessarily normal to the buoy axis at P'' was considered to require no correction to the lateral force, and only a small one to the torque. The reasoning is indicated in connection with the diagram shown below. The integration over all sections up to P'' includes the set of (negative pressure) forces F_1 which does not actually act, and fails to include the set F_2 which does. F_1 and F_2 are taken to have equal resultants, so that the only effect of replacing one by the other is a small torque depending on their difference in axial positions.



Results: The differential equations of motion can be written

$$\frac{d^2}{dt^2}\,\xi = -g\,(a/L)\,h \tag{5}$$

$$(d^2\eta/dt^2) + \omega_V^2\,\eta = \omega_V^2\,\eta_1 - (ga/L)\,v \tag{6}$$

and

$$(d^2\theta/dt^2) + \omega_\theta^2 \sin\theta = (-gr/LR^2)(\sin\theta)\,T_1 + (ga/LR^2)\,T_2 + (gr/LR^2)\,T_3 \tag{7}$$

in which

$$\begin{aligned} h &= r\cos(\alpha+\theta) + (1-r)\,e^{-k\cos\theta(\ell_1+\delta+\epsilon)}\cos(\alpha+\theta+k\sin\theta\{\ell_1+\delta+\epsilon\}) - e^{-k\cos\theta(\ell_2+\delta+\epsilon)}\cos(\alpha+\theta+k\sin\theta\{\ell_2+\delta+\epsilon\}) \\ &\approx [\,r + (1-r)\,e^{-k\ell_1} - e^{-k\ell_2}\,]\cos\alpha \end{aligned} \tag{8}$$

$$\omega_V^2 = (gr/L) \qquad \omega_\theta^2 = g\,(\ell_g - \ell_b)\,/\,R^2 \tag{9}$$

$$\begin{aligned} \eta_1 &= (\ell_g - \eta)\,(\sec\theta - 1) - a\tan\theta\cos(\alpha+\theta) \\ &\approx (1/2)\,\ell_g\,\theta^2 - a\,\theta\cos\alpha \end{aligned} \tag{10}$$

$$\begin{aligned} v &= (1-r)\,e^{-k\cos\theta(\ell_1+\delta+\epsilon)}\sin(\alpha+\theta+k\sin\theta\{\ell_1+\delta+\epsilon\}) - e^{-k\cos\theta(\ell_2+\delta+\epsilon)}\sin(\alpha+\theta+k\sin\theta\{\ell_2+\delta+\epsilon\}) \\ &\approx [\,(1-r)\,e^{-k\ell_1} - e^{-k\ell_2}\,]\sin\alpha \end{aligned} \tag{11}$$

$$\begin{aligned} T_1 &= (\ell_g + \tfrac{1}{2}\delta)\,\delta - \tfrac{1}{2}\,\epsilon^2 \\ &\approx -\ell_g\eta + \tfrac{1}{2}[\,(\ell\theta)^2 + \eta^2 - (a\sin\alpha)^2\,] \end{aligned} \tag{12}$$

$$T_2 = r\,[\,(\ell_g + \delta + \varepsilon)\ \sec\theta - 1/K\,]\cos(\alpha+\theta) + (1-r)e^{-k\cos\theta(\ell_1+\delta+\varepsilon)}$$

$$[\,(\ell_g - \ell_1)\cos(\alpha + k\sin\theta\{\ell_1+\delta+\varepsilon\}) - 1/k\cos(\alpha+\theta+k\sin\theta\{\ell_1+\delta+\varepsilon\})\,]$$

$$+ e^{-k\cos\theta(\ell_2+\delta+\varepsilon)}[\,(\ell_2 - \ell_g)\cos(\alpha + k\sin\theta\{\ell_2+\delta+\varepsilon\})$$

$$+ 1/k\cos(\alpha+\theta+k\sin\theta\{\ell_2+\delta+\varepsilon\})\,)]$$

$$\simeq [\,r(\ell_g - 1/k) + (1-r)e^{-k\ell_1}(\ell_g - \ell_1 - 1/k) + e^{-k\ell_2}(\ell_2 - \ell_g + 1/k)]\cos\alpha$$

$$= (\ell_g - \ell_{\theta})\,h \tag{13}$$

$$T_3 = (1/8\pi)\,r\,A\sin(\emptyset-\theta)\tan^2(\emptyset-\theta) \quad \simeq \quad 0 \tag{14}$$

In all of the expressions (8) through (14) there is given first an instantaneous value as accurately as it was calculated. This is followed by a suitable simplification in which some small terms are dropped, and others altered to be applicable to a whole cycle of motion, assumed periodic with fundamental frequency ω. In the simplified expressions α must be understood to refer to the mean position of P" throughout a cycle.

It is evident that to a good approximation the three coordinates may be considered separately. By (5), ξ may be considered as a forced oscillation of a free particle.

η may be considered as the motion of an oscillator of natural frequency ω_V under a periodic impressed force represented by the term containing v in (6). The terms in η_1 are quadratic and hence tend to contribute small constant displacements and second harmonic distortion.

Similarly the motion in θ is largely governed by the natural frequency ω_θ and the periodic impressed torque represented by the term containing T_2 in (7). The term involving T_3 arises from lack of perpendicularity between the buoy axis and the sea surface, and is quite

small. A positive constant part of T_1 represents increased restoring torque and a consequent rise in natural frequency. The periodic parts of T_1 contribute constant displacement, or second or third harmonic distortion. These perturbations will all generally be small.

The first-order steady state motions are

$$\xi = (a/kL)\,h \tag{15}$$

$$\eta = [\, ga/L\,(\omega^2 - \omega_V^2)\,]\, v$$

$$= av \,/\, (kL - r) \tag{16}$$

$$\theta = \{ (\ell_g - \ell_\theta) \,/\, [\, R^2 - \frac{1}{k}\,(\ell_g - \ell_b)\,]\,\}(a/kL)h \tag{17}$$

The horizontal amplitude approaches a for small k (and ω), and is considerably less than a at higher frequencies. The vertical amplitude has a resonant peak when $kL = r$ and a zero when $1 - r = \ell^{-k(\ell_2 - \ell_1)}$. The zero will always occur at a higher frequency than the resonance. The vertical amplitude also approaches a at low frequency, and vanishes exponentially at high frequency.

The distance ℓ_θ defined in (13) locates the effective center of horizontal wave force relative to point P. Its value varies from zero at very high frequencies to ℓ_b at very low frequencies. The relation (17) shows directly the relation between angular displacement and linear displacement at the center of mass. The center of oscillation can be varied over quite wide limits by choice of the resonant frequency ω_θ.

Effects of drag: The velocities indicated above for the buoy are usually in phase with (but sometimes differ by 180° phase from) the water wave velocities. In either case the drag forces will add energy to the motion where the water velocity is in phase with, and greater than, the buoy velocity; otherwise damping will occur.

The following order-of-magnitude estimates of the ratio of drag to pressure-plus-gravity force have been made on the assumption that the buoy is stationary. For the horizontal motion, assuming

unity drag coefficient for a cylindrical form against transverse flow, the ratio at any frequency is of the order a/D, where D is the diameter of the cylinder. For end-drag due to vertical flow, at any frequency the ratio is of the order of ka, again assuming unity drag coefficient. For skin-drag arising from vertical flow, the ratio at high frequency is of the order (ac) / (Dv) where c is the skin-drag coefficient, with a possible value in the range 0.002 to 0.003. At low frequencies, the ratio is less than that given by this expression. This result indicates a lower limit for the vertical pressure excitation, below which it will be superseded by skin friction excitation.

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